Lecture 31: Hertzian Dipole Antenna.

In the previous lecture, we discussed the fundamental source-field relationship that can be used to calculate the $\vec{E}$ and $\vec{H}$ fields produced by sinusoidal steady state line currents. In that process, we first compute the phasor vector magnetic potential

$$\vec{A}(\vec{r}) = \int e^\mu \vec{I}(\vec{r}') e^{-j\beta R} \frac{dl'}{4\pi R}$$

where $R = |\vec{r} - \vec{r}'|$, then compute the phasor magnetic field as

$$\vec{H}(\vec{r}) = \frac{1}{\mu} \nabla \times \vec{A}(\vec{r})$$

Using Ampere’s law, we determine the phasor electric field

$$\vec{E}(\vec{r}) = \frac{1}{j\omega \epsilon} \nabla \times \vec{H}(\vec{r})$$

What might be surprising to you is that one of the most difficult challenges an antenna designer faces is determining the current distribution $\vec{I}(\vec{r}')$ on a particular antenna. This is often a complicated problem and usually the only accurate solution is to use computational electromagnetics methods.

However, there is a famous class of straight wire antenna that has an accurate approximate solution for the form of the current on the antenna. This class of antenna is known as a thin-wire dipole antenna:
It is called a “dipole” antenna because there are two wires or poles comprising the antenna.

The form of the current on the wire is known quite accurately for thin wires ($a \ll L$) with length $L$ that is less than approximately $1 \lambda$. The currents are less accurately known for longer lengths.

We will only be considering the simplest dipole antenna in this course. It is called the Hertzian dipole, or point dipole, antenna:

It is infinitesimally short with a uniform current distribution. Because

$$i(t) = \frac{dq(t)}{dt}$$

then for sinusoidal steady state this means that $I = j\omega Q$. 
The ends of this short antenna are capacitively loaded by small metallic spheres or disks to create a current that is approximately uniform along its length in Fig. 2.

Without the capacitive loading the current would be approximately triangularly shaped with a maximum value at the center of the antenna and linearly decreasing to zero at both ends, as illustrated in Fig. 3:

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**Short Dipole Antenna Current Magnitudes:**

<table>
<thead>
<tr>
<th>With End Loading</th>
<th>Without End Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
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**$\vec{A}$ for a Hertzian Dipole Antenna**

Following the procedure described at the beginning of page 1, we’ll now determine the $\vec{E}$ and $\vec{H}$ fields produced (i.e., “radiated”) by the Hertzian dipole antenna. This three step process begins with the calculation of $\vec{A}$ from (1)

$$\vec{A}(\vec{r}) = \int_{\mathcal{C}} \mu \bar{I}(\vec{r'}) e^{-j\beta R} \frac{dl'}{4\pi R}$$  \hspace{1cm} (1)
The current \( \vec{I}(r') \) is that for the Hertzian dipole antenna, which we will assume is located at the center of a spherical coordinate system:

Substituting for this assumed uniform current into (1) gives

\[
\vec{A}(\vec{r}) = \int_{-\Delta L/2}^{\Delta L/2} \hat{a}_z \mu I \frac{e^{-j\beta R}}{4\pi R} dz' = \hat{a}_z \mu I \frac{\Delta L/2}{4\pi} \int_{-\Delta L/2}^{\Delta L/2} \frac{e^{-j\beta R}}{R} dz' \tag{4}
\]

Because this antenna is infinitesimally short then \( R \) is approximately constant over the entire range of integration. Consequently, the integral in (4) can be performed trivially leading to

\[
\vec{A}(\vec{r}) = \hat{a}_z \frac{\mu I \Delta L}{4\pi} \frac{e^{-j\beta r}}{r} \text{ [Wb/m]} \tag{5}
\]

(Notice that \( R \) has now become \( r \).)

So we now have completed the first step and have obtained an analytical expression for \( \vec{A} \). The next step is the calculation of \( \vec{H} \). Because of the \( \phi \) spherical symmetry of the dipole antenna,
we’ll use the spherical coordinate system to simplify the mathematics.

We can convert $\vec{A}$ in (5) to spherical vector components quite easily by using the relationship

$$\hat{a}_z = \hat{a}_r \cos \theta - \hat{a}_\theta \sin \theta$$

(6)

giving

$$\vec{A}(\vec{r}) = \hat{a}_r A_r + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi^0$$

(7)

where

$$A_r = A_z \cos \theta = \frac{\mu I \Delta L}{4\pi} \cos \theta \frac{e^{-j\beta r}}{r}$$

(8)

$$A_\theta = -A_z \sin \theta = -\frac{\mu I \Delta L}{4\pi} \sin \theta \frac{e^{-j\beta r}}{r}$$

(9)

$$A_\phi = 0$$

(10)

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**$\vec{H}$ for a Hertzian Dipole Antenna**

Because we have successfully determined $\vec{A}$ produced by the Hertzian dipole antenna, we can now determine the $\vec{E}$ and $\vec{H}$ fields, beginning with $\vec{H}$. As we saw in (2)

$$\vec{H}(\vec{r}) = \frac{1}{\mu} \nabla \times \vec{A}(\vec{r})$$

(2)

Because of the $\phi$ symmetry, $\partial/\partial \phi \to 0$ which, using the determinant form of curl in the spherical coordinate system, gives
\[
\bar{H}(\bar{r}) = \frac{1}{\mu r^2 \sin \theta} \begin{vmatrix}
\hat{a}_r & \hat{a}_\theta r & \hat{a}_\phi r \sin \theta \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\
A_r & rA_\theta & 0
\end{vmatrix}
\]

such that

\[
\bar{H}(\bar{r}) = \frac{1}{\mu r^2 \sin \theta} \left\{ \hat{a}_\phi r \sin \theta \left[ \frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \right\}
\]

\[
= \hat{a}_\phi \frac{1}{\mu r} \left[ \frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right]
\]

- Using (9)
\[
\frac{\partial}{\partial r} (rA_\theta) = \frac{\partial}{\partial r} \left( -r \frac{\mu I \Delta L}{4\pi} \sin \theta \frac{e^{-j\beta r}}{r} \right) = j\beta \frac{\mu I \Delta L}{4\pi} \sin \theta e^{-j\beta r} \quad (12)
\]

- Using (8)
\[
\frac{\partial A_r}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \frac{\mu I \Delta L}{4\pi} \cos \theta \frac{e^{-j\beta r}}{r} \right) = -\frac{\mu I \Delta L}{4\pi} \sin \theta \frac{e^{-j\beta r}}{r} \quad (13)
\]

Consequently, substituting (12) and (13) in (11), we find \( \bar{H} \) to be

\[
\bar{H}(\bar{r}) = \hat{a}_\phi \frac{1}{\mu r} \left( j\beta \frac{\mu I \Delta L}{4\pi} \sin \theta e^{-j\beta r} + \frac{\mu I \Delta L}{4\pi} \sin \theta \frac{e^{-j\beta r}}{r} \right)
\]

\[
= \hat{a}_\phi \frac{I \Delta L}{4\pi} \sin \theta \left( \frac{j\beta}{r} + \frac{1}{r^2} \right) e^{-j\beta r}
\]
Factoring out \((j\beta)^2 = -\beta^2\) from both terms in (14) we arrive at the final form of the magnetic field produced (or “radiated”) by the Hertzian dipole antenna

\[
\vec{H}(\vec{r}) = -\hat{a}_\phi \frac{I\Delta L}{4\pi} \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} \right] e^{-j\beta r} \text{ [A/m]} \tag{15}
\]

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**\(\vec{E}\) for a Hertzian Dipole Antenna**

Now that \(\vec{H}\) has been determined, we can solve for the \(\vec{E}\) field produced by this Hertzian dipole antenna according to (3). Using the determinant form of curl in the spherical coordinate system

\[
\vec{E}(\vec{r}) = \frac{1}{j\omega \varepsilon} \nabla \times \vec{H} = \frac{1}{j\omega \varepsilon} \cdot \frac{1}{r^2 \sin \theta} \begin{vmatrix}
\hat{a}_r & \hat{a}_\theta r & \hat{a}_\phi r \sin \theta \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\
0 & 0 & r \sin \theta H_\phi
\end{vmatrix}
\]

such that

\[
\vec{E}(\vec{r}) = \frac{1}{j\omega \varepsilon \cdot r^2 \sin \theta} \left[ \hat{a}_r \frac{\partial}{\partial \theta} (r \sin \theta H_\phi) - \hat{a}_\theta r \frac{\partial}{\partial r} (r \sin \theta H_\phi) \right] = \frac{1}{j\omega \varepsilon} \left[ \hat{a}_r \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta H_\phi) - \hat{a}_\theta \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \right] \tag{16}
\]

Substituting for \(H_\phi\) from (15) into (16), and simplifying, gives

\[
\vec{E}(\vec{r}) = \hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi \text{ [V/m]} \tag{17}
\]
where
\[
E_r = -\frac{I\Delta L}{4\pi} \eta \beta^2 2 \cos \theta \left[ \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}
\]  
(18)

\[
E_\theta = -\frac{I\Delta L}{4\pi} \eta \beta^2 \sin \theta \left[ \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2} + \frac{1}{(j\beta r)^3} \right] e^{-j\beta r}
\]  
(19)

\[
E_\phi = 0
\]  
(20)

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**Summary**

So this concludes the calculation of the $E$ and $H$ fields produced (or “radiated”) by the Hertzian dipole antenna. There is a wealth of information contained in these field solutions for $E$ in (17)-(20) and $H$ in (15) that we will carefully pick through in the next lecture.