But

\[ I_{\text{enc}} = \oint_S \mathbf{J} \cdot d\mathbf{S} \]  \hfill (7.18)

Comparing the surface integrals in eqs. (7.17) and (7.18) clearly reveals that

\[ \nabla \times \mathbf{H} = \mathbf{J} \]  \hfill (7.19)

This is the third Maxwell’s equation to be derived; it is essentially Ampere’s law in differential (or point) form whereas eq. (7.16) is the integral form. From eq. (7.19), we should observe that \( \nabla \times \mathbf{H} = \mathbf{J} \neq 0 \); that is, magnetostatic field is not conservative.

### 7.4 APPLICATIONS OF AMPERE’S LAW

We now apply Ampere’s circuit law to determine \( \mathbf{H} \) for some symmetrical current distributions as we did for Gauss’s law. We will consider an infinite line current, an infinite current sheet, and an infinitely long coaxial transmission line.

#### A. Infinite Line Current

Consider an infinitely long filamentary current \( I \) along the \( z \)-axis as in Figure 7.10. To determine \( \mathbf{H} \) at an observation point \( P \), we allow a closed path pass through \( P \). This path, on which Ampere’s law is to be applied, is known as an Amperian path (analogous to the term Gaussian surface). We choose a concentric circle as the Amperian path in view of eq. (7.14), which shows that \( \mathbf{H} \) is constant provided \( \rho \) is constant. Since this path encloses the whole current \( I \), according to Ampere’s law

\[ I = \int H_{\phi} a_{\phi} \cdot \rho \ d\phi \ a_{\phi} = H_{\phi} \int \rho d\phi = H_{\phi} \cdot 2\pi \rho \]

![Figure 7.10 Ampere’s law applied to an infinite filamentary line current.](image)
as expected from eq. (7.14).

**B. Infinite Sheet of Current**

Consider an infinite current sheet in the \( z = 0 \) plane. If the sheet has a uniform current density \( K = K_y a_y \) A/m as shown in Figure 7.11, applying Ampere’s law to the rectangular closed path (Amperian path) gives

\[
\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b
\]  

(7.21a)

To evaluate the integral, we first need to have an idea of what \( \mathbf{H} \) is like. To achieve this, we regard the infinite sheet as comprising of filaments; \( d\mathbf{H} \) above or below the sheet due to a pair of filamentary currents can be found using eqs. (7.14) and (7.15). As evident in Figure 7.11(b), the resultant \( d\mathbf{H} \) has only an \( x \)-component. Also, \( \mathbf{H} \) on one side of the sheet is the negative of that on the other side. Due to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of \( \mathbf{H} \) for a pair are the same for the infinite current sheets, that is,

\[
\mathbf{H} = \begin{cases} 
H_y a_x & z > 0 \\
-H_y a_x & z < 0
\end{cases}
\]  

(7.21b)

**Figure 7.11** Application of Ampere’s law to an infinite sheet: (a) closed path 1-2-3-4-1, (b) symmetrical pair of current filaments with current along \( a_y \).
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where \( H_o \) is yet to be determined. Evaluating the line integral of \( \mathbf{H} \) in eq. (7.21b) along the closed path in Figure 7.11(a) gives

\[
\oint \mathbf{H} \cdot d\mathbf{l} = \left( \int_{1}^{2} + \int_{3}^{4} + \int_{4}^{1} \right) \mathbf{H} \cdot d\mathbf{l} = 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) = 2H_o b
\]

(7.21c)

From eqs. (7.21a) and (7.21c), we obtain \( H_o = \frac{1}{2} K_y \). Substituting \( H_o \) in eq. (7.21b) gives

\[
\mathbf{H} = \begin{cases} 
\frac{1}{2} K_y \mathbf{a}_y, & z > 0 \\
-\frac{1}{2} K_y \mathbf{a}_y, & z < 0 
\end{cases} 
\]

(7.22)

In general, for an infinite sheet of current density \( \mathbf{K} \) A/m,

\[
\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n
\]

(7.23)

where \( \mathbf{a}_n \) is a unit normal vector directed from the current sheet to the point of interest.

C. Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The cross section of the line is shown in Figure 7.12, where the z-axis is out of the page. The inner conductor has radius \( a \) and carries current \( I \) while the outer conductor has inner radius \( b \) and thickness \( t \) and carries return current \( -I \). We want to determine \( \mathbf{H} \) everywhere assuming that current is uniformly distributed in both conductors. Since the current distribution is symmetrical, we apply Ampere's law along the Amperian paths.

Figure 7.12 Cross section of the transmission line; the positive z-direction is out of the page.
perian path for each of the four possible regions: \(0 \leq \rho \leq a\), \(a \leq \rho \leq b\), \(b \leq \rho \leq b + t\), and \(\rho \geq b + t\).

For region \(0 \leq \rho \leq a\), we apply Ampere’s law to path \(L_1\), giving

\[
\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = \int J \cdot d\mathbf{S}
\]  

(7.24)

Since the current is uniformly distributed over the cross section,

\[
J = \frac{I}{\pi a^2} \mathbf{a}_z, \quad d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z
\]

\[
I_{\text{enc}} = \int J \cdot d\mathbf{S} = \frac{I}{\pi a^2} \int \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2 = \frac{I \rho^2}{a^2}
\]

Hence eq. (7.24) becomes

\[
H_\phi \int dl = H_\phi 2\pi \rho = \frac{I \rho^2}{a^2}
\]

or

\[
H_\phi = \frac{I \rho}{2\pi a^2}
\]  

(7.25)

For region \(a \leq \rho \leq b\), we use path \(L_2\) as the Amperian path,

\[
\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = I
\]

\[
H_\phi 2\pi \rho = I
\]

or

\[
H_\phi = \frac{I}{2\pi \rho}
\]  

(7.26)

since the whole current \(I\) is enclosed by \(L_2\). Notice that eq. (7.26) is the same as eq. (7.14) and it is independent of \(a\). For region \(b \leq \rho \leq b + t\), we use path \(L_3\), getting

\[
\oint_{L_3} \mathbf{H} \cdot d\mathbf{l} = H_\phi \cdot 2\pi \phi = I_{\text{enc}}
\]  

(7.27a)

where

\[
I_{\text{enc}} = I + \int J \cdot d\mathbf{S}
\]
and $\mathbf{J}$ in this case is the current density (current per unit area) of the outer conductor and is along $-\mathbf{a}_z$, that is,

$$\mathbf{J} = \frac{I}{\pi[(b + t)^2 - b^2]} \mathbf{a}_z$$

Thus

$$I_{\text{enc}} = I - \frac{I}{\pi[(b + t)^2 - b^2]} \int_{\phi=0}^{\phi=\rho} \rho \, d\rho \, d\phi$$

$$= I \left[ 1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right]$$

Substituting this in eq. (7.27a), we have

$$H_\phi = \frac{I}{2\pi \rho} \left[ 1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right]$$

(7.27b)

For region $\rho \geq b + t$, we use path $L_4$, getting

$$\oint_{L_4} \mathbf{H} \cdot d\mathbf{l} = I - I = 0$$

or

$$H_\phi = 0$$

(7.28)

Putting eqs. (7.25) to (7.28) together gives

$$\mathbf{H} = \begin{cases} 
\frac{I \rho}{2\pi a^2} \mathbf{a}_\phi, & 0 \leq \rho \leq a \\
\frac{I}{2\pi \rho} \mathbf{a}_\phi, & a \leq \rho \leq b \\
\frac{I}{2\pi \rho} \left[ 1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] \mathbf{a}_\phi, & b \leq \rho \leq b + t \\
0, & \rho \geq b + t
\end{cases}$$

(7.29)

The magnitude of $\mathbf{H}$ is sketched in Figure 7.13.

Notice from these examples that the ability to take $\mathbf{H}$ from under the integral sign is the key to using Ampere's law to determine $\mathbf{H}$. In other words, Ampere's law can only be used to find $\mathbf{H}$ due to symmetric current distributions for which it is possible to find a closed path over which $\mathbf{H}$ is constant in magnitude.
7.4 APPLICATIONS OF AMPERE'S LAW  

Figure 7.13 Plot of $H_\phi$ against $\rho$.

**EXAMPLE 7.5**

Planes $z = 0$ and $z = 4$ carry current $K = -10a_x A/m$ and $K = 10a_x A/m$, respectively. Determine $H$ at

(a) $(1, 1, 1)$

(b) $(0, -3, 10)$

**Solution:**

Let the parallel current sheets be as in Figure 7.14. Also let

$$H = H_0 + H_4$$

where $H_0$ and $H_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively. We make use of eq. (7.23).

(a) At $(1, 1, 1)$, which is between the plates ($0 < z = 1 < 4$),

$$H_0 = \frac{1}{2} K \times a_n = \frac{1}{2} (-10a_x) \times a_z = 5a_y A/m$$

$$H_4 = \frac{1}{2} K \times a_n = \frac{1}{2} (10a_x) \times (-a_z) = 5a_y A/m$$

Hence,

$$H = 10a_y A/m$$

Figure 7.14 For Example 7.5; parallel infinite current sheets.